PROCEEDINGS OF THE ROYAL SOCIETY A

FirstCite e-publishing

Proc. R. Soc. A
doi:10.1098/rspa.2010.0125
Published online

A long-wave model for the surface elastic wave in a coated half-space

By H.-H. Dai[1], J. Kaplunov[2] and D. A. Prikazchikov[3,*]

[1]*Department of Mathematics, City University of Hong Kong, Kowloon Tong, Kowloon, Hong Kong*
[2]*Department of Mathematical Sciences, SISCM, Brunel University, Uxbridge, UB8 3PH, UK*
[3]*Department of Computational Mathematics and Mathematical Physics, Bauman Moscow State Technical University, Moscow, Russia*

The paper deals with the three-dimensional problem in linear isotropic elasticity for a coated half-space. The coating is modelled via the effective boundary conditions on the surface of the substrate initially established on the basis of an ad hoc approach and justified in the paper at a long-wave limit. An explicit model is derived for the surface wave using the perturbation technique, along with the theory of harmonic functions and Radon transform. The model consists of three-dimensional 'quasi-static' elliptic equations over the interior subject to the boundary conditions on the surface which involve relations expressing wave potentials through each other as well as a two-dimensional hyperbolic equation singularly perturbed by a pseudo-differential (or integro-differential) operator. The latter equation governs dispersive surface wave propagation, whereas the elliptic equations describe spatial decay of displacements and stresses. As an illustration, the dynamic response is calculated for impulse and moving surface loads. The explicit analytical solutions obtained for these cases may be used for the non-destructive testing of the thickness of the coating and the elastic moduli of the substrate.

Keywords: asymptotic model; surface wave; coating; Radon transform; singular perturbation

1. Introduction

The theory of elastic surface waves for coated bodies is highly important for many modern applications, for example, thin-film technology. As examples of ongoing interest to the problem, we mention recent publications (Fu 2007; Steigmann & Ogden 2007; Qian *et al.* 2009 and references therein).

*Author for correspondence (prikazchikovda@yandex.ru).

Received 2 March 2010
Accepted 19 April 2010

The potential of the exact analysis in the framework of three-dimensional elasticity is rather limited due to a sophisticated dispersion relation (e.g. Achenbach & Keshava 1967). This motivates the development of approximate formulations including modelling the effect of the coating by introducing certain effective boundary conditions imposed on the substrate surface (Auld 1990 and references therein). In particular, such conditions were derived in Tiersten (1969) within the isotropic context using physical assumptions based on the classical theory of plate extensions. Later on, it was claimed by Bövik (1996) that the results of Tiersten (1969) are not asymptotically consistent in the case of a thin coating. A standard asymptotic procedure presented in §3 of the paper justifies the consistency of the consideration in Tiersten (1969). Moreover, as shown in §5, the dispersion relation associated with the aforementioned boundary conditions in Tiersten (1969) coincides with the two-term long-wave expansion of the exact dispersion relation studied in Shuvalov & Every (2008).

To our best knowledge, very little, if any, explicit asymptotic results have been obtained for the transient dynamic response of a coated half-space subject to surface loading. A way forward is to extract from the wave equations in linear elasticity an approximate model oriented for the surface wave phenomena only. A two-dimensional version of the model was established in Kaplunov *et al.* (2006) by perturbing the original plane strain equations about the homogeneous Rayleigh wave solution expressed in terms of harmonic functions (Friedlander 1948; Chadwick 1976). The resulting relations consist of 'quasi-static' elliptic equations over the interior for the wave Lame potentials satisfying at the surface the 'dynamic' boundary conditions involving an inhomogeneous hyperbolic equation. These clearly indicate a dual elliptic–hyperbolic nature of the Rayleigh wave. A high accuracy of the predictions of the model in Kaplunov *et al.* (2006) was demonstrated in Kaplunov *et al.* (in press) by comparison with the exact solution of the transient moving load problem. We also remark that the philosophy underlying the approach in Kaplunov *et al.* (2006, in press) seems to be analogous to that in Achenbach (1998) and Parker & Kiselev (2009) dealing with specialized formulations for the free surface waves of general profile.

In §4, we extend the results of Kaplunov *et al.* (2006) in two directions. First, we incorporate the effect of a coating via the effective boundary conditions leading to a weak long-wave dispersion of the surface wave. In contrast to an uncoated half-space, the hyperbolic equation for the surface now contains a singular perturbation in the form of an integro-differential or a pseudo-differential operator (see discussion in §5). Second, we proceed to a three-dimensional problem starting from the Radon transform that has been traditionally used for the three-dimensional to two-dimensional reduction in linear elasticity (e.g. Willis 1971; Wang & Achenbach 1996; Georgiadis & Lykotrafitis 2001).

In §6, the proposed model is applied to plane problems for surface-impulsive and moving loads. As might be expected, the behaviour of the phase velocity in the vicinity of its maximum or minimum corresponding to the long-wave Rayleigh limit determines the advancing or receding type of the surface wave front induced by a point instantaneous impulse. This behaviour also affects the resonant response in case of a moving load.

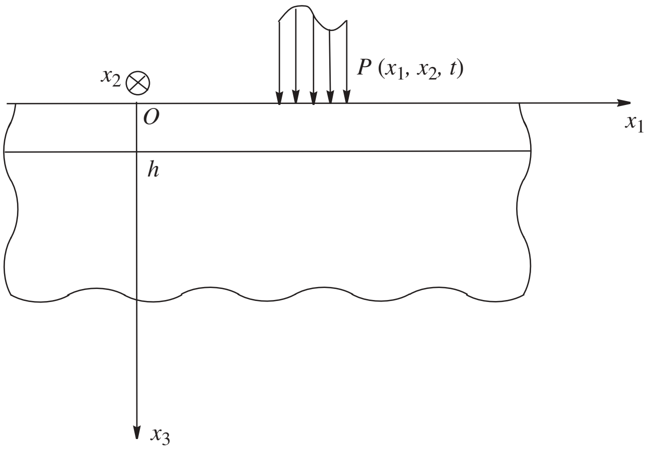

Figure 1. Statement of the problem.

2. Statement of the problem

We consider a three-dimensional half-space covered by an elastic coating of constant thickness h. The Cartesian axes Ox_1 and Ox_2 are lying on the upper surface of the coating, with the interface between the coating and the half-space given by $x_3 = h$ (figure 1).

The equations of motion in three-dimensional elasticity are written as

$$\left.\begin{aligned} \frac{\partial\sigma_{ii}}{\partial x_i} + \frac{\partial\sigma_{ij}}{\partial x_j} + \frac{\partial\sigma_{i3}}{\partial x_3} &= \rho\frac{\partial^2 u_i}{\partial t^2} \\ \text{and} \qquad \frac{\partial\sigma_{3i}}{\partial x_i} + \frac{\partial\sigma_{3j}}{\partial x_j} + \frac{\partial\sigma_{33}}{\partial x_3} &= \rho\frac{\partial^2 u_3}{\partial t^2}, \end{aligned}\right\} \tag{2.1}$$

where u_n are the components of displacement vector, σ_{in}, σ_{3n} are the components of the Cauchy stress tensor and ρ is the volume density of solid media. We suppose, throughout the paper, unless otherwise stated, that $i \neq j = 1, 2$ and $n = 1, 2, 3$. The constitutive relations for a linear isotropic elastic solid are given by

$$\left.\begin{aligned} &\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \sigma_{ii} = (\lambda + 2\mu)\frac{\partial u_i}{\partial x_i} + \lambda\left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_3}{\partial x_3}\right) \\ \text{and} \quad &\sigma_{3i} = \sigma_{i3} = \mu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right), \qquad \sigma_{33} = \lambda\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3}, \end{aligned}\right\} \tag{2.2}$$

where λ and μ are elastic moduli. The associated bulk wave speeds may then be expressed as

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad c_2 = \sqrt{\frac{\mu}{\rho}}. \tag{2.3}$$

We start from the relations (2.1) and (2.2) to describe the substrate ($x_3 \geq h$). In the case of the coating ($0 \leq x_3 \leq h$), we supply with suffix '0' the material parameters in the above equations, operating with ρ_0, λ_0, μ_0 as well as c_{10} and c_{20}.

The prescribed boundary conditions at the surface $x_3 = 0$ of the coating are

$$\sigma_{i3} = 0 \quad \text{and} \quad \sigma_{33} = -P, \tag{2.4}$$

where $P = P(x_1, x_2, t)$ is a normal force. We also assume continuity of all displacements and stresses σ_{n3} at the interface $x_3 = h$.

The aim of the consideration below is to establish a long-wave model for the surface wave propagating in the system 'coating-substrate'. The derivation contains two stages. First, we reduce the effect of the coating on the substrate by effective boundary conditions at the interface. Then, we perturb the equations for the substrate subject to these boundary conditions about the homogeneous surface wave solution.

3. Effective boundary conditions

In this section, we take into separate consideration the problem for an elastic coating, setting the following boundary conditions at the interface $x_3 = h$:

$$u_n = v_n, \tag{3.1}$$

where $v_n = v_n(x_1, x_2, t)$ are prescribed displacements and $n = 1, 2, 3$ as above.

We apply to the boundary value problem for the coating (2.1), (2.2), (2.4) and (3.1) the standard scheme of direct asymptotic integration (e.g. Goldenveizer *et al.* 1993 and references therein). Let us specify a small parameter

$$\varepsilon = \frac{h}{L} \ll 1 \tag{3.2}$$

corresponding to the long-wave assumption, where L is a typical wave length, and scale the original variables as

$$\xi_i = \frac{x_i}{L}, \qquad \eta = \frac{x_3}{h} \quad \text{and} \quad \tau = \frac{tc_{20}}{L}. \tag{3.3}$$

We also define the dimensionless quantities

$$\left.\begin{aligned} &u_n^* = \frac{1}{V}\,u_n, \qquad v_n^* = \frac{1}{V} v_n, \\ \text{and} \qquad &\sigma_{ij}^* = \frac{L}{\mu_0 V}\sigma_{ij}, \qquad \sigma_{n3}^* = \frac{L^2}{\mu_0 h V}\sigma_{n3}, \qquad p^* = \frac{L^2}{\mu_0 h V} P, \end{aligned}\right\} \tag{3.4}$$

where V is the maximal displacement amplitude. Here and below in this section all quantities with the asterisk are of the same asymptotic order.

The equations of motion and the constitutive relations may now be rewritten as

$$\left.\begin{aligned}
&\frac{\partial\sigma^*_{ii}}{\partial\xi_i}+\frac{\partial\sigma^*_{ij}}{\partial\xi_j}+\frac{\partial\sigma^*_{i3}}{\partial\eta}=\frac{\partial^2u^*_i}{\partial\tau^2},\\
&\frac{\partial\sigma^*_{33}}{\partial\eta}+\varepsilon\left(\frac{\partial\sigma^*_{i3}}{\partial\xi_i}+\frac{\partial\sigma^*_{j3}}{\partial\xi_j}\right)=\frac{\partial^2u^*_3}{\partial\tau^2},\\
&\sigma^*_{ij}=\frac{\partial u^*_i}{\partial\xi_j}+\frac{\partial u^*_j}{\partial\xi_i},\\
&\varepsilon\sigma^*_{ii}=\left(\kappa_0^2-2\right)\frac{\partial u^*_3}{\partial\eta}+\varepsilon\left(\kappa_0^2\frac{\partial u^*_i}{\partial\xi_i}+\left(\kappa_0^2-2\right)\frac{\partial u^*_j}{\partial\xi_j}\right),\\
&\varepsilon^2\sigma^*_{i3}=\frac{\partial u^*_i}{\partial\eta}+\varepsilon\frac{\partial u^*_3}{\partial\xi_i}\\
\text{and}\qquad&\varepsilon^2\sigma^*_{33}=\kappa_0^2\frac{\partial u^*_3}{\partial\eta}+\varepsilon\left(\kappa_0^2-2\right)\left(\frac{\partial u^*_i}{\partial\xi_i}+\frac{\partial u^*_j}{\partial\xi_j}\right),
\end{aligned}\right\}\tag{3.5}$$

where $\kappa_0=c_{10}/c_{20}$.

The boundary conditions become

$$\left.\begin{aligned}
&\sigma^*_{i3}=0\quad\text{and}\quad\sigma^*_{33}=-p^*\quad\text{at }\eta=0,\\
\text{and}\qquad&u^*_n=v^*_n\quad\text{at }\eta=1.
\end{aligned}\right\}\tag{3.6}$$

Next we expand the displacements and stresses in asymptotic series in terms of the small parameter ε:

$$\begin{pmatrix}u^*_n\\ \sigma^*_{ii}\\ \sigma^*_{ij}\\ \sigma^*_{3i}\\ \sigma^*_{33}\end{pmatrix}=\begin{pmatrix}u^{(0)}_n\\ \sigma^{(0)}_{ii}\\ \sigma^{(0)}_{ij}\\ \sigma^{(0)}_{3i}\\ \sigma^{(0)}_{33}\end{pmatrix}+\varepsilon\begin{pmatrix}u^{(1)}_k\\ \sigma^{(1)}_{ii}\\ \sigma^{(1)}_{ij}\\ \sigma^{(1)}_{3i}\\ \sigma^{(1)}_{33}\end{pmatrix}+\cdots.\tag{3.7}$$

Substituting these expansions into the equations (3.5), and boundary conditions (3.6), we obtain at the leading order

$$\left.\begin{aligned}
&\frac{\partial\sigma^{(0)}_{ii}}{\partial\xi_i}+\frac{\partial\sigma^{(0)}_{ij}}{\partial\xi_j}+\frac{\partial\sigma^{(0)}_{i3}}{\partial\eta}=\frac{\partial^2u^{(0)}_i}{\partial\tau^2},\\
&\frac{\partial\sigma^{(0)}_{33}}{\partial\eta}=\frac{\partial^2u^{(0)}_3}{\partial\tau^2},\\
&\sigma^{(0)}_{ij}=\frac{\partial u^{(0)}_i}{\partial\xi_j}+\frac{\partial u^{(0)}_j}{\partial\xi_i},\\
&\frac{\partial u^{(0)}_n}{\partial\eta}=0,\\
&u^{(0)}_n=v^*_n\quad\text{at }\eta=0,\\
&\sigma^{(0)}_{33}=-p^*\quad\text{at }\eta=0\\
\text{and}\qquad&\sigma^{(0)}_{i3}=0\quad\text{at }\eta=0.
\end{aligned}\right\}\tag{3.8}$$

Equations $(3.8)_2$ and $(3.8)_4$ together with the related boundary conditions $(3.8)_5$ and $(3.8)_6$ may be integrated to yield

$$u_n^{(0)} = v_n^* \tag{3.9}$$

and

$$\sigma_{33}^{(0)} = \eta \frac{\partial^2 v_3^*}{\partial \tau^2} - p^*. \tag{3.10}$$

The last formula represents the sought for expression for the normal stress σ_{33}. Evaluation of the stresses σ_{i3} requires further analysis.

At the next order, equation $(3.5)_5$ becomes

$$\frac{\partial u_i^{(1)}}{\partial \eta} + \frac{\partial u_3^{(0)}}{\partial \xi_i} = 0. \tag{3.11}$$

It is also clear that

$$u_n^{(1)} = 0 \quad \text{at } \eta = 1. \tag{3.12}$$

Then, from equations (3.11) and (3.12), we have

$$u_i^{(1)} = (1-\eta) \frac{\partial v_3^*}{\partial \xi_i}. \tag{3.13}$$

Similarly, we derive from the formula $(3.5)_6$ taken at order ε, and the boundary conditions (3.12)

$$u_3^{(1)} = (1 - 2\kappa_0^{-2})(1-\eta) \left(\frac{\partial v_i^*}{\partial \xi_i} + \frac{\partial v_j^*}{\partial \xi_j} \right). \tag{3.14}$$

Now, using the formulae $(3.5)_4$, (3.9) and (3.14), we obtain

$$\begin{aligned} \sigma_{ii}^{(0)} &= \kappa_0^2 \frac{\partial u_i^{(0)}}{\partial \xi_i} + (\kappa_0^2 - 2) \frac{\partial u_j^{(0)}}{\partial \xi_j} + (\kappa_0^2 - 2) \frac{\partial u_3^{(1)}}{\partial \eta} \\ &= 4(1-\kappa_0^{-2}) \frac{\partial v_i^*}{\partial \xi_i} + 2(1 - 2\kappa_0^{-2}) \frac{\partial v_j^*}{\partial \xi_j}. \end{aligned} \tag{3.15}$$

Finally, starting from equation $(3.8)_1$ and using equations $(3.8)_3$ and (3.15) with the boundary conditions $(3.8)_7$, we are in a position to determine the leading order terms $\sigma_{i3}^{(0)}$ in the asymptotic expansions for the stresses σ_{i3}^*. They are

$$\sigma_{i3}^{(0)} = \eta \left[\frac{\partial^2 v_i^*}{\partial \tau^2} - \frac{\partial^2 v_i^*}{\partial \xi_j^2} - 4(1-\kappa_0^{-2}) \frac{\partial^2 v_i^*}{\partial \xi_i^2} - (3 - 4\kappa_0^{-2}) \frac{\partial^2 v_j^*}{\partial \xi_i \partial \xi_j} \right]. \tag{3.16}$$

In the original dimension form, the stresses at the interface $x_3 = h$ may be presented as

$$\left. \begin{aligned} s_{i3} &= \rho_0 h \left\{ \frac{\partial^2 v_i}{\partial t^2} - c_{20}^2 \left[\frac{\partial^2 v_i}{\partial x_j^2} + 4(1-\kappa_0^{-2}) \frac{\partial^2 v_i}{\partial x_i^2} + (3 - 4\kappa_0^{-2}) \frac{\partial^2 v_j}{\partial x_i \partial x_j} \right] \right\} \\ \text{and} \quad s_{33} &= \rho_0 h \frac{\partial^2 v_3}{\partial t^2} - P, \end{aligned} \right\} \tag{3.17}$$

where $s_{n3} = \sigma_{n3}(x_1, x_2, h, t)$. The assumed continuity of the displacements and stresses at $x_3 = h$ now reads as $\boldsymbol{\sigma}_{n3} = s_{n3}$ and $u_n = v_n$, where $\boldsymbol{\sigma}_{n3}$ and u_n denote the stresses and displacements in the substrate, respectively. As a result, the effective boundary conditions on the surface of the substrate can be written as

$$\left.\begin{aligned} \sigma_{i3} &= \rho_0 h \left\{ \frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left[\frac{\partial^2 u_i}{\partial x_j^2} + 4(1-\kappa_0^{-2})\frac{\partial^2 u_i}{\partial x_i^2} + (3-4\kappa_0^{-2})\frac{\partial^2 u_j}{\partial x_i \partial x_j} \right] \right\} \\ \text{and} \quad \sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} - P. \end{aligned}\right\} \tag{3.18}$$

These boundary conditions in case of $P = 0$ coincide with those obtained in Tiersten (1969) using an ad hoc physical approach based on modelling of the coating in the framework of the classical theory of plate extension. A more recent analysis in Bövik (1996), with the methodology later extended to anisotropy in, for example, Niklasson *et al.* (2000), claims that boundary conditions (3.18) neglect a few terms of $O(h)$.

Let us consider expressions (35) and (36) from Bövik (1996) in the notation of the present paper.

$$\left.\begin{aligned} \sigma_{i3} &= \rho_0 h \left\{ \frac{\partial^2 u_i}{\partial t^2} - c_{20}^2 \left[\frac{\partial^2 u_i}{\partial x_j^2} + 4(1-\kappa_0^{-2})\frac{\partial^2 u_i}{\partial x_i^2} + (3-4\kappa_0^{-2})\frac{\partial^2 u_j}{\partial x_i \partial x_j} \right] \right\} \\ &\quad \underline{- h(1-2\kappa_0^{-2})\frac{\partial \sigma_{33}}{\partial x_i}} \\ \text{and} \quad \sigma_{33} &= \rho_0 h \frac{\partial^2 u_3}{\partial t^2} \underline{- h\left(\frac{\partial \sigma_{3i}}{\partial x_i} + \frac{\partial \sigma_{3j}}{\partial x_j}\right)}. \end{aligned}\right\} \tag{3.19}$$

The underlined terms in the r.h.s. of the above formulae do not appear in the effective boundary conditions (3.18). It can easily be verified by introducing in equations (3.19) the scaled variables (3.3) and dimensionless displacements and stresses (3.4) that all the extra terms are of the next asymptotic order in ε, that is, $O(h^2)$ in the notation of Bövik (1996).

4. Derivation of the asymptotic model

In this section, we investigate the problem for the substrate governing equations (2.1) and (2.2) over the domain $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$ and $h < x_3 < \infty$, subject to effective boundary conditions (3.18) at the surface $x_3 = h$.

First, we express the equations of motion in terms of displacements. On substituting equation (2.2) into equation (2.1), we obtain

$$(\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu \Delta \boldsymbol{u} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{4.1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector, and a conventional notation is adopted for three-dimensional differential operators.

Let us apply the Radon transforms to equation (4.1) (Georgiadis & Lykotrafitis 2001 for more details). The result is

$$
\left.
\begin{aligned}
&[(\lambda+\mu)\cos^2\alpha+\mu]\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_1^{(\alpha)}}{\partial x_3^2}\\
&\quad+(\lambda+\mu)\cos\alpha\left(\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}\right)=\rho\frac{\partial^2 u_1^{(\alpha)}}{\partial t^2},\\
&[(\lambda+\mu)\sin^2\alpha+\mu]\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_2^{(\alpha)}}{\partial x_3^2}\\
&\quad+(\lambda+\mu)\sin\alpha\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi^2}+\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}\right)=\rho\frac{\partial^2 u_2^{(\alpha)}}{\partial t^2}\\
\text{and}\quad&(\lambda+\mu)\left(\cos\alpha\frac{\partial^2 u_1^{(\alpha)}}{\partial\chi\partial x_3}+\sin\alpha\frac{\partial^2 u_2^{(\alpha)}}{\partial\chi\partial x_3}\right)+\mu\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi^2}\\
&\quad+(\lambda+2\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial x_3^2}=\rho\frac{\partial^2 u_3^{(\alpha)}}{\partial t^2},
\end{aligned}
\right\}
\tag{4.2}
$$

where

$$
u_k^{(\alpha)}(\chi,\alpha,x_3,t)=\int_{-\infty}^{\infty}u_k(\chi\cos\alpha-\zeta\sin\alpha,\chi\sin\alpha+\zeta\cos\alpha,x_3,t)\,d\zeta, \tag{4.3}
$$

and

$$
\chi=x_1\cos\alpha+x_2\sin\alpha,\quad \zeta=-x_1\sin\alpha+x_2\cos\alpha,
$$

with the angle α varying over the interval $0\le\alpha<2\pi$. We also introduce the transformed displacements in the Cartesian frame (χ,ζ), (figure 2). They are

$$
u_\chi^{(\alpha)}=u_1^{(\alpha)}\cos\alpha+u_2^{(\alpha)}\sin\alpha \quad\text{and}\quad u_\zeta^{(\alpha)}=-u_1^{(\alpha)}\sin\alpha+u_2^{(\alpha)}\cos\alpha. \tag{4.4}
$$

Now we set $u_\zeta^{(\alpha)}=0$ assuming that the anti-plane motion is not induced by the studied normal force. Then equations (4.2) rewritten in terms of the displacements (4.4) take the form of the plane problem of elasticity, that is,

$$
\left.
\begin{aligned}
&(\lambda+2\mu)\frac{\partial^2 u_\chi^{(\alpha)}}{\partial\chi^2}+\mu\frac{\partial^2 u_\chi^{(\alpha)}}{\partial x_3^2}+(\lambda+\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi\partial x_3}=\rho\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2}\\
\text{and}\quad&(\lambda+\mu)\frac{\partial^2 u_\chi^{(\alpha)}}{\partial\chi\partial x_3}+\mu\frac{\partial^2 u_3^{(\alpha)}}{\partial\chi^2}+(\lambda+2\mu)\frac{\partial^2 u_3^{(\alpha)}}{\partial x_3^2}=\rho\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2}.
\end{aligned}
\right\}
\tag{4.5}
$$

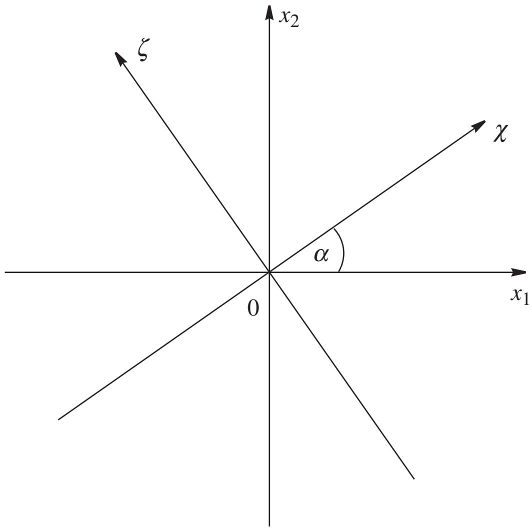

Figure 2. Rotation of Cartesian frame.

The associated boundary conditions at the surface $x_3 = h$ follow from the transformation of the original boundary conditions (3.18). They can be written as

$$\left.\begin{aligned} \sigma_{\chi 3}^{(\alpha)} &= \mu\left(\frac{\partial u_\chi^{(\alpha)}}{\partial x_3} + \frac{\partial u_3^{(\alpha)}}{\partial \chi}\right) = \mu_0 h\left[\frac{\partial^2 u_\chi^{(\alpha)}}{\partial t^2} c_{20}^{-2} - 4(1-\kappa_0^{-2})\frac{\partial^2 u_\chi^{(\alpha)}}{\partial \chi^2}\right] \\ \text{and}\quad \sigma_{33}^{(\alpha)} &= \lambda\frac{\partial u_\chi^{(\alpha)}}{\partial \chi} + (\lambda+2\mu)\frac{\partial u_3^{(\alpha)}}{\partial x_3} = \mu_0 h \frac{\partial^2 u_3^{(\alpha)}}{\partial t^2} c_{20}^{-2} - P^{(\alpha)}, \end{aligned}\right\} \tag{4.6}$$

where $\sigma_{\chi 3}^{(\alpha)}, \sigma_{33}^{(\alpha)}$ and $P^{(\alpha)}$ denote the transformed stresses and normal force, respectively.

In case of $h = 0$, the boundary value problems (4.5) and (4.6) are identical to those considered in Kaplunov *et al.* (2006), dealing with an explicit asymptotic model for the plane Rayleigh waves induced by the transient loads applied to the surface of an elastic half-space. Below we extend the asymptotic methodology in Kaplunov *et al.* (2006) to a coated half-space.

Let us introduce the transformed wave potentials $\phi^{(\alpha)}$ and $\psi^{(\alpha)}$ by

$$u_\chi^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial \chi} - \frac{\partial \psi^{(\alpha)}}{\partial x_3} \quad \text{and} \quad u_3^{(\alpha)} = \frac{\partial \phi^{(\alpha)}}{\partial x_3} + \frac{\partial \psi^{(\alpha)}}{\partial \chi}. \tag{4.7}$$

On inserting the latter into equations (4.5) and (4.6), we obtain

$$\left.\begin{aligned} \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_1^2}\frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} &= 0, \\ \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} + \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} - \frac{1}{c_2^2}\frac{\partial^2 \psi^{(\alpha)}}{\partial t^2} &= 0, \end{aligned}\right\} \tag{4.8}$$

and

$$\left.\begin{aligned}
&\mu\left[2\frac{\partial^2\phi^{(\alpha)}}{\partial\chi\partial x_3}+\frac{\partial^2\psi^{(\alpha)}}{\partial\chi^2}-\frac{\partial^2\psi^{(\alpha)}}{\partial x_3^2}\right]=\mu_0 h\left[c_{20}^{-2}\left(\frac{\partial^3\phi^{(\alpha)}}{\partial\chi\partial t^2}-\frac{\partial^3\psi^{(\alpha)}}{\partial x_3\partial t^2}\right)\right.\\
&\qquad\left.-4(1-\kappa_0^{-2})\left(\frac{\partial^3\phi^{(\alpha)}}{\partial\chi^3}-\frac{\partial^3\psi^{(\alpha)}}{\partial x_3\partial\chi^2}\right)\right]\\
&\mu\left[(\kappa^2-2)\frac{\partial^2\phi^{(\alpha)}}{\partial\chi^2}+\kappa^2\frac{\partial^2\phi^{(\alpha)}}{\partial x_3^2}+2\frac{\partial^2\psi^{(\alpha)}}{\partial\chi\partial x_3}\right]\\
&\qquad=\mu_0 h c_{20}^{-2}\left(\frac{\partial^3\phi^{(\alpha)}}{\partial x_3\partial t^2}+\frac{\partial^3\psi^{(\alpha)}}{\partial\chi\partial t^2}\right)-P^{(\alpha)}.
\end{aligned}\right\}\tag{4.9}$$

We begin with the scaling

$$\xi=\frac{\chi-c_R t}{L},\qquad \gamma=\frac{x_3-h}{L}\quad\text{and}\quad \tau=\frac{c_R\varepsilon}{L}t,\tag{4.10}$$

where $\varepsilon=h/L\ll 1$, L is a typical wavelength and c_R denotes a surface wave speed, which is not supposed yet to be the classical Rayleigh one. The parameter ε here states a small deviation of the phase velocities of interest from the wave speed c_R.

The wave equations (4.8) are expressed in variables (4.10) as

$$\left.\begin{aligned}
&\frac{\partial^2\phi^{(\alpha)}}{\partial\gamma^2}+k_1^2\frac{\partial^2\phi^{(\alpha)}}{\partial\xi^2}+2\varepsilon(1-k_1^2)\frac{\partial^2\phi^{(\alpha)}}{\partial\xi\partial\tau}-\varepsilon^2(1-k_1^2)\frac{\partial^2\phi^{(\alpha)}}{\partial\tau^2}=0\\
\text{and}\quad&\frac{\partial^2\psi^{(\alpha)}}{\partial\gamma^2}+k_2^2\frac{\partial^2\psi^{(\alpha)}}{\partial\xi^2}+2\varepsilon(1-k_2^2)\frac{\partial^2\psi^{(\alpha)}}{\partial\xi\partial\tau}-\varepsilon^2(1-k_2^2)\frac{\partial^2\psi^{(\alpha)}}{\partial\tau^2}=0,
\end{aligned}\right\}\tag{4.11}$$

where

$$k_i^2=1-\frac{c_R^2}{c_i^2}.$$

The two-term asymptotic solution of equation (4.11) is given by Kaplunov *et al.* (2006)

$$\left.\begin{aligned}
&\phi^{(\alpha)}=\frac{P_0L^3}{\mu\,\varepsilon}(\phi^{(0)}(\xi,\gamma,\tau)+\varepsilon\phi^{(1)}(\xi,\gamma,\tau))\\
\text{and}\quad&\psi^{(\alpha)}=\frac{P_0L^3}{\mu\,\varepsilon}(\psi^{(0)}(\xi,\gamma,\tau)+\varepsilon\psi^{(1)}(\xi,\gamma,\tau)),
\end{aligned}\right\}\tag{4.12}$$

where P_0 is the maximal amplitude of the normal force P, $\phi^{(0)}=\phi^{(0)}(\xi,k_1\gamma,\tau)$ and $\psi^{(0)}=\psi^{(0)}(\xi,k_2\gamma,\tau)$ are arbitrary plane harmonic functions in the first two arguments and

$$\phi^{(1)}=\phi^{(1,0)}+\gamma\phi^{(1,1)}\quad\text{and}\quad\psi^{(1)}=\psi^{(1,0)}+\gamma\psi^{(1,1)},\tag{4.13}$$

where $\phi^{(1,0)} = \phi^{(1,0)}(\xi, k_1\gamma, \tau)$ and $\psi^{(1,0)} = \psi^{(1,0)}(\xi, k_2\gamma, \tau)$ are again plane harmonic functions and

$$\phi^{(1,1)} = -\frac{1-k_1^2}{k_1}\frac{\partial\bar{\phi}^{(0)}}{\partial\tau} \quad \text{and} \quad \psi^{(1,1)} = -\frac{1-k_2^2}{k_2}\frac{\partial\bar{\psi}^{(0)}}{\partial\tau} \tag{4.14}$$

with bar denoting a harmonic conjugate.

On substituting the two-term asymptotic behaviours (4.12) into the boundary conditions (4.9), we have at the leading order

$$\left.\begin{aligned} &2\frac{\partial^2\phi^{(0)}}{\partial\xi\partial\gamma} + \frac{\partial^2\psi^{(0)}}{\partial\xi^2} - \frac{\partial^2\psi^{(0)}}{\partial\gamma^2} = 0 \\ \text{and} \quad &(\kappa^2-2)\frac{\partial^2\phi^{(0)}}{\partial\xi^2} + \kappa^2\frac{\partial^2\phi^{(0)}}{\partial\gamma^2} + 2\frac{\partial^2\psi^{(0)}}{\partial\xi\partial\gamma} = 0. \end{aligned}\right\} \tag{4.15}$$

Next we exploit the Cauchy–Riemann identities for a plane harmonic function $f(x, ky)$

$$\frac{\partial f}{\partial y} = -k\frac{\partial\bar{f}}{\partial x}, \qquad \frac{\partial f}{\partial x} = \frac{1}{k}\frac{\partial\bar{f}}{\partial y}, \quad \bar{\bar{f}} = -f, \tag{4.16}$$

having from equations (4.15)

$$\left.\begin{aligned} &2k_1\frac{\partial^2\phi^{(0)}}{\partial\xi^2} + (1+k_2^2)\frac{\partial^2\bar{\psi}^{(0)}}{\partial\xi^2} = 0 \\ \text{and} \quad &(1+k_2^2)\frac{\partial^2\phi^{(0)}}{\partial\xi^2} + 2k_2\frac{\partial^2\bar{\psi}^{(0)}}{\partial\xi^2} = 0. \end{aligned}\right\} \tag{4.17}$$

The compatibility of equation (4.17) leads to the classical Rayleigh surface wave equation presented as

$$4k_1k_2 - (1+k_2^2)^2 = 0. \tag{4.18}$$

Thus, the wave speed c_R introduced in equations (4.10) is precisely the conventional Rayleigh wave speed. The boundary conditions (4.17) also result in the relation

$$\bar{\psi}^{(0)} = -\frac{2k_1}{1+k_2^2}\phi^{(0)} \quad \text{at } \gamma = 0 \tag{4.19}$$

between the transformed potentials at the surface.

At the next order, the boundary conditions (4.9) become

$$\left.\begin{aligned} &2\frac{\partial^2\phi^{(1,0)}}{\partial\xi\partial\gamma} + \frac{\partial^2\psi^{(1,0)}}{\partial\xi^2} - \frac{\partial^2\psi^{(1,0)}}{\partial\gamma^2} - \frac{2(1-k_1^2)}{k_1}\frac{\partial^2\bar{\phi}^{(0)}}{\partial\xi\partial\tau} + \frac{2(1-k_2^2)}{k_2}\frac{\partial^2\bar{\psi}^{(0)}}{\partial\gamma\partial\tau} \\ &\qquad + m(k_{20}^2 + 3 - 4\kappa_0^{-2})\left(\frac{\partial^3\phi^{(0)}}{\partial\xi^3} - \frac{\partial^3\bar{\psi}^{(0)}}{\partial\gamma\partial\xi^2}\right) = 0 \\ \text{and} \quad &(\kappa^2-2)\frac{\partial^2\phi^{(1,0)}}{\partial\xi^2} + \kappa^2\frac{\partial^2\phi^{(1,0)}}{\partial\gamma^2} + 2\frac{\partial^2\psi^{(1,0)}}{\partial\xi\partial\gamma} + \frac{2\kappa^2(1-k_1^2)}{k_1}\frac{\partial^2\bar{\phi}^{(0)}}{\partial\gamma\partial\tau} \\ &\qquad - \frac{2(1-k_2^2)}{k_2}\frac{\partial^2\bar{\psi}^{(0)}}{\partial\xi\partial\tau} - m(1-k_{20}^2)\left(\frac{\partial^3\phi^{(0)}}{\partial\xi^2\partial\gamma} - \frac{\partial^3\bar{\psi}^{(0)}}{\partial\xi^3}\right) = -\frac{P^{(\alpha)}}{P_0L}, \end{aligned}\right\} \tag{4.20}$$

where

$$k_{20}^2 = 1 - \frac{c_R^2}{c_{20}^2} \quad \text{and} \quad m = \frac{\mu_0}{\mu}.$$

After a fairly straightforward algebra making use of the Cauchy–Riemann identities and the formula (4.19), we obtain from equations (4.20) the following:

$$\left.\begin{aligned} & 2k_1 \frac{\partial^2 \phi^{(1,0)}}{\partial \xi^2} + (1+k_2^2)\frac{\partial^2 \bar{\psi}^{(1,0)}}{\partial \xi^2} + 2\left[\frac{1-k_1^2}{k_1} - \frac{2(1-k_2^2)k_1}{1+k_2^2}\right]\frac{\partial^2 \phi^{(0)}}{\partial \xi \partial \tau} \\ & \qquad + \frac{m}{2}(k_{20}^2 + 3 - 4\kappa_0^2)(1-k_2^2)\frac{\partial^3 \bar{\phi}^{(0)}}{\partial \xi^3} = 0 \\ \text{and} \quad & (1+k_2^2)\frac{\partial^2 \phi^{(1,0)}}{\partial \xi^2} + 2k_2 \frac{\partial^2 \bar{\psi}^{(1,0)}}{\partial \xi^2} + 2(1-k_2^2)\left[1 - \frac{2k_1}{(1+k_2^2)k_2}\right]\frac{\partial^2 \phi^{(0)}}{\partial \xi \partial \tau} \\ & \qquad - mk_1(1-k_{20}^2)\frac{1-k_2^2}{1+k_2^2}\frac{\partial^3 \bar{\phi}^{(0)}}{\partial \xi^3} = \frac{P^{(\alpha)}}{P_0 L}. \end{aligned}\right\} \tag{4.21}$$

Finally, we eliminate $\partial^2 \bar{\psi}^{(1,0)}/\partial \xi^2$ from these equations and perform algebraic manipulations involving the Rayleigh equation (4.18). The result is

$$2\frac{\partial^2 \phi^{(0)}}{\partial \xi \partial \tau} + \frac{b}{k_1}\frac{\partial^3 \phi^{(0)}}{\partial \gamma \partial \xi^2} = \frac{(1+k_2^2)P^{(\alpha)}}{2BP_0L} \quad \text{at } \gamma = 0, \tag{4.22}$$

where

$$B = (1-k_1^2)\frac{k_2}{k_1} + (1-k_2^2)\frac{k_1}{k_2} - (1-k_2^4)$$

and

$$b = \frac{m}{2B}(1-k_2^2)\left[(1-k_{20}^2)(k_1+k_2) - 4k_2(1-\kappa_0^{-2})\right]. \tag{4.23}$$

Now we suppose that in equations (4.12)

$$\phi^{(\alpha)} = \frac{P_0 L^3}{\mu\, \varepsilon}\phi^{(0)} \quad \text{and} \quad \psi^{(\alpha)} = \frac{P_0 L^3}{\mu\, \varepsilon}\psi^{(0)}. \tag{4.24}$$

Then, equation (4.22) may be re-cast in terms of the dimensional variables (χ, x_3, t) as

$$\frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} - \frac{1}{c_R^2}\frac{\partial^2 \phi^{(\alpha)}}{\partial t^2} + \frac{bh}{k_1}\frac{\partial^3 \phi^{(\alpha)}}{\partial \chi^2 \partial x_3} = AP^{(\alpha)} \quad \text{at } x_3 = h, \tag{4.25}$$

with $A = (1+k_2^2)/2\mu B$.

Similarly, at the leading order, the transformed potentials over the interior satisfy the elliptic equations obtained from equations (4.11), that is,

$$\left.\begin{aligned} & \frac{\partial^2 \phi^{(\alpha)}}{\partial x_3^2} + k_1^2 \frac{\partial^2 \phi^{(\alpha)}}{\partial \chi^2} = 0 \\ \text{and} \quad & \frac{\partial^2 \psi^{(\alpha)}}{\partial x_3^2} + k_2^2 \frac{\partial^2 \psi^{(\alpha)}}{\partial \chi^2} = 0, \end{aligned}\right\} \tag{4.26}$$

and the potential $\psi^{(\alpha)}$ at the surface is determined from the relation (4.19) in the form

$$\frac{\partial \psi^{(\alpha)}}{\partial \chi} = -\frac{2}{1+k_2^2}\frac{\partial \phi^{(\alpha)}}{\partial x_3} \quad \text{at } x_3 = h. \tag{4.27}$$

In what follows it is convenient to operate with a pair of the potentials $\psi_1^{(\alpha)} = \psi^{(\alpha)}\cos\alpha$ and $\psi_2^{(\alpha)} = \psi^{(\alpha)}\sin\alpha$ along with the potential ϕ. By inverting Radon transform in (4.25)–(4.27), we arrive at the long-wave model given by the elliptic equations over the interior $(x_3 \geq h)$

$$\frac{\partial^2 \phi}{\partial x_3^2} + k_1^2 \varDelta_2 \phi = 0 \tag{4.28}$$

and

$$\frac{\partial^2 \psi_i}{\partial x_3^2} + k_2^2 \varDelta_2 \psi_i = 0, \tag{4.29}$$

with the boundary conditions at $x_3 = h$

$$\varDelta_2 \phi - \frac{1}{c_R^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{bh}{k_1}\frac{\partial}{\partial x_3}(\varDelta_2 \phi) = AP \tag{4.30}$$

and

$$\frac{\partial \psi_i}{\partial x_i} = -\frac{2}{1+k_2^2}\frac{\partial \phi}{\partial x_3}, \tag{4.31}$$

where $\varDelta_2$ denotes two-dimensional Laplacian in the coordinates (x_1, x_2).

In this case, the displacements are expressed through the potentials ϕ and ψ_i as (see equations (4.4) and (4.7))

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi_1}{\partial x_3}, \quad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi_2}{\partial x_3}, \quad \text{and} \quad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2}. \tag{4.32}$$

5. Discussion of the model

The asymptotic model derived in the previous section is oriented to the surface wave only, ignoring the effect of the bulk waves. The contribution of the surface wave is dominant for a number of practically important problems including, in particular, the resonance phenomena caused by moving loads (e.g. Kaplunov *et al.* in press). The model includes the three-dimensional elliptic equations (4.28) and (4.29) governing the spatial decay over the interior along with the conditions (4.30) and (4.31) imposed on the surface. The second of them relates the potentials ψ_1 and ψ_2 with the potential ϕ, whereas the first one is the only relation in the model involving the dynamic factor; in doing so, it operates with the component of the original load P inducing the surface wave. In the absence of a coating $(h = 0)$, this reduces to an inhomogeneous two-dimensional wave equation generalizing the elliptic–hyperbolic formulation in Kaplunov *et al.* (2006) for the plane strain problem.

The presence of a coating leads to a coupling between the elliptic equation (4.28) and the singularly perturbed hyperbolic equation (4.30) due to the normal derivative with respect to x_3 entering the latter. To a certain extent, the coupled problem (4.28) and (4.30) may be interpreted as a membrane resting on an incompressible fluid.

Although the scaling (4.10) underlying the model assumes that the effect of coating has to be taken into consideration at the leading order approximation, it is obvious that the equation on the surface is valid for a weak coupling as well. In this case, the dynamic response of a coated half-space can be calculated using a relatively simple regular perturbation scheme.

It is well known that the equation (4.30) can be reduced to the form of an integro-differential or pseudo-differential equation on the surface. To this end, we have to express the normal derivative $\partial\phi/\partial x_3$ in terms of the potential at the surface $\phi(x_1, x_2, 0, t)$. In particular, the solution of equation (4.28) can be rewritten in a symbolic form as

$$\phi(x_1, x_2, x_3, t) = \exp\left(-k_1\sqrt{-\varDelta_2}x_3\right)\phi(x_1, x_2, 0, t), \tag{5.1}$$

where $\sqrt{-\varDelta_2}$ is understood as a pseudo-differential operator. Then, we have

$$\left.\frac{\partial\phi}{\partial x_3}\right|_{x_3=0} = -k_1\sqrt{-\varDelta_2}\phi(x_1, x_2, 0, t) \tag{5.2}$$

and equation (4.30) becomes

$$\varDelta_2\phi - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} - bh\sqrt{-\varDelta_2}\varDelta_2\phi = AP. \tag{5.3}$$

In case of a plane strain problem (say if $\partial\phi/\partial x_2 = 0$), the last equation reduces to a one-dimensional pseudo-differential equation. It is

$$\frac{\partial^2\phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} - bh\sqrt{-\partial_1^2}\frac{\partial^2\phi}{\partial x_1^2} = AP. \tag{5.4}$$

Similar to Kovalev *et al.* (2002), equation (5.3) can also be presented as an integro-differential equation. Using the identities

$$\sqrt{-\partial_1^2}\phi = -\frac{1}{k_1}\frac{\partial\phi}{\partial x_3} = \frac{\partial\bar{\phi}}{\partial x_1} = H\frac{\partial\phi}{\partial x_1} \quad \text{at } x_3 = 0, \tag{5.5}$$

where $H(f)(x)$ denotes the Hilbert transform, given by

$$H(f)(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(z)}{x-z}\mathrm{d}z, \tag{5.6}$$

we have

$$\frac{\partial^2\phi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2\phi}{\partial t^2} - bhH\frac{\partial^3\phi}{\partial x_1^3} = AP. \tag{5.7}$$

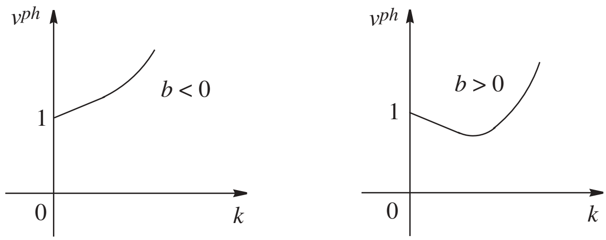

Figure 3. Typical long-wave behaviour.

The dispersion relation associated with the homogeneous equations (5.4) and (5.7) is

$$v^{ph} = 1 - \frac{b}{2}|k| + \dots, \tag{5.8}$$

where k is the wave number normalized by h^{-1} and v^{ph} is the phase speed normalized by c_R.

The above expression coincides with the long-wave asymptotic expansion of the 'exact' dispersion relation analysed by Shuvalov & Every (2008). In the special case in which $\mu_0 = \mu$ and $\lambda_0 = \lambda$, it reduces to the dispersion relation displayed in Kovalev *et al.* (2002) with

$$b = \frac{c_R^2(\rho_0 - \rho)(1 - k_2^2)(k_1 + k_2)}{2\mu B}. \tag{5.9}$$

These comparisons provide one more illustration of the asymptotic consistency of the effective boundary conditions in Tiersten (1969) demonstrated in §3.

It is also worth mentioning that, as it has been shown in Shuvalov & Every (2008), the parameter b in the proposed model may take both positive and negative values corresponding to the local minimum and maximum of the phase velocity equal to the Rayleigh wave speed, (figure 3 and also formula (5.8)). As might be expected, the sign of b is crucial for the surface dynamic behaviour see examples in §6.

6. Illustrative examples

In this section, we evaluate the dynamic response of a coated half-space within the framework of the long-wave model for two particular types of surface loading, namely for an instantaneous point impulse and also for a distributed moving load. For the sake of simplicity, we deal with plane strain problems.

(*a*) *Impulse loading*

Let us set $P = P_0\delta(x_1)\delta(t)$ in the r.h.s. of equation (5.3) and specify dimensionless variables by

$$r = \frac{x_1}{L} \quad \text{and} \quad \tau = \frac{c_R t}{L}, \tag{6.1}$$

where L is a chosen linear scale. Then, equation (5.4) becomes

$$\frac{\partial^2\theta}{\partial r^2} - \frac{\partial^2\theta}{\partial \tau^2} - h_L \text{ sign } b\sqrt{-\partial_r^2}\frac{\partial^2\theta}{\partial r^2} = -\delta(r)\delta(\tau), \tag{6.2}$$

where $\theta = -2\phi/Ac_R P_0$ and the parameter $h_L = h|b|/L$ is assumed to be small, that is, $h_L \ll 1$.

Below we develop the scheme in Emri *et al.* (2001) of the method of matched asymptotic expansions. We begin with the boundary layer at the characteristic of the degenerated equation (6.2). The inner co-ordinate is

$$\zeta = \frac{\tau - r}{\sqrt{h_L}}, \tag{6.3}$$

indicating that the width of the boundary layer is proportional to $h^{1/2}$.

On substituting the scaling (6.3) into the homogeneous equation (6.2), we obtain at leading order

$$\frac{\partial\theta^{\text{inn}}}{\partial\tau} - \frac{h_L}{2} \text{ sign } b\sqrt{-\partial_\zeta^2}\frac{\partial\theta^{\text{inn}}}{\partial\zeta} = 0. \tag{6.4}$$

The solution of the latter can be presented in the form of a Fourier integral as

$$\theta^{\text{inn}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(C(\omega)\exp\left[\mathrm{i}\omega\left(\zeta_0 - \frac{h_L|\omega|\tau \text{ sign } b}{2}\right)\right]\right)\mathrm{d}\omega, \tag{6.5}$$

where $\zeta_0 = \tau - r$ and arbitrary function $C(\omega)$ has to follow from matching with the appropriate outer expansion.

In the outer region, equation (6.2) reduces to the one-dimensional wave equation

$$\frac{\partial^2\theta^{\text{out}}}{\partial r^2} - \frac{\partial^2\theta^{\text{out}}}{\partial\tau^2} = -2\delta(r)\delta(\tau), \tag{6.6}$$

from which we have $\theta^{\text{out}} = H(\zeta_0)$, or, identically,

$$\theta^{\text{out}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right)\exp[\mathrm{i}\omega\zeta_0]\,\mathrm{d}\omega. \tag{6.7}$$

Matching of the expansions (6.5) and (6.7) yields for the former (see also Cole 1968 for more detail)

$$\theta^{\text{inn}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\delta(\omega) + \frac{1}{\mathrm{i}\omega}\right)\exp\left[\mathrm{i}\omega\left(\zeta_0 - \frac{h_L\omega\tau \text{ sign } b}{2}\right)\right]\mathrm{d}\omega, \tag{6.8}$$

resulting in the uniform asymptotic behaviour

$$\theta = \frac{1}{2} - \frac{1}{\pi}\text{sign}\, b\, I\left(\frac{(r-\tau)\text{ sign } b}{\sqrt{2h_L\tau}}\right), \tag{6.9}$$

where (Prudnikov *et al.* 1986)

$$I(x) = \int_0^{\infty}\frac{\sin(t^2 + 2tx)}{t}\,\mathrm{d}t = \frac{\pi}{2}\left[\frac{1}{2} + \text{sign}\, x[C(x) + S(x)] - C^2(x) - S^2(x)\right], \tag{6.10}$$

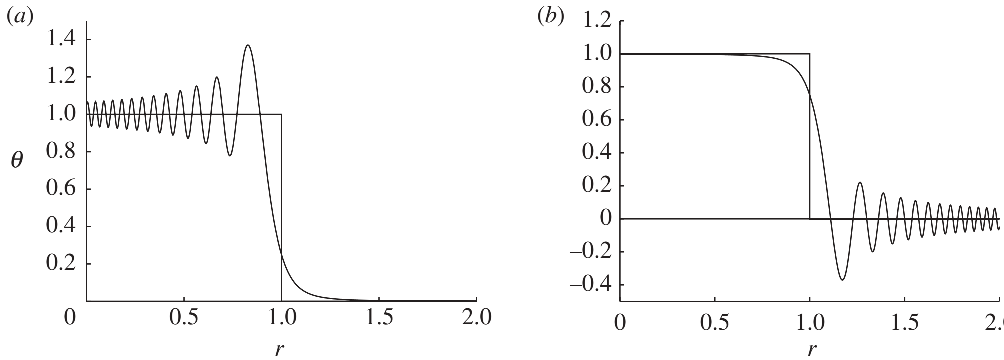

Figure 4. Receding and advancing fronts.

and $C(x)$ and $S(x)$ are Fresnel integrals

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) \mathrm{d}t \quad \text{and} \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) \mathrm{d}t.$$

The obtained solution, due to the restrictions of the surface wave model, does not take into account the slowly varying component of the overall wave field corresponding to the bulk waves. The latter becomes negligible in the case of the derivatives of the function ϕ in equation ((6.9)). For an applied force P, equation (6.9) provides the analytical expression of the wave profile for given c_R, bh and A (cf. equations (5.4) and (5.7)). On the other hand, if for an applied P one measures the wave profile and compares it with equation (6.9), c_R, bh and A can then be determined. Thus, the analytical results presented here may be used for the non-destructive testing of the coating thickness h and the Lamé constants λ and μ of the substrate for the known elastic moduli of the coating.

Numerical results are displayed in figure 4 for $\tau = 1$ and $h_L = 0.01$. In case $b > 0$ corresponding to the local maximum of the phase velocity at the Rayleigh wave speed (figure 3*a*, we observe a receding front. Case $b < 0$ (figure 3*b*) is related to the minimum of the phase velocity. As a result, we arrive at an advancing front.

We also remark that the receding and advancing fronts of plate extension were studied in Dai & Cai (1999, 2000) and Kaplunov *et al.* (2000) for pre-stressed structures. Although the solution of Kaplunov *et al.* (2000) is expressed through the Airy functions, the graphs in the cited paper look pretty similar to those given in figure 4.

(*b*) *Steady-state moving load problem*

Now, we consider the effect of a distributed load steadily moving along the surface with the constant speed c. Taking the load in the form

$$P = \frac{P_0 l}{\pi[l^2 + (x_1 - ct)^2]},$$

where the parameter l determines the load distribution, we obtain from equation (5.4)

$$g\sigma - h_l\sqrt{-\partial_s^2}\sigma = \frac{1}{1 + s^2}, \tag{6.11}$$

with

$$\sigma = \frac{AP_0}{\pi l}\frac{\partial^2 \phi}{\partial s^2}, \quad s = \frac{x_1 - ct}{l}, \quad g = 1 - \frac{c^2}{c_R^2}, \quad h_l = \frac{bh}{l}.$$

where σ corresponds to the normal surface stress, s is the dimensionless moving co-ordinate and g and h_l are the key problem parameters characterizing the thickness of the coating and the proximity of the speed of the load to the Rayleigh wave speed, respectively.

Next, using the Fourier transform, we have

$$\sigma = \int_0^\infty \frac{e^{-\omega}\cos(\omega s)}{g - h_l \omega} d\omega. \tag{6.12}$$

In this paper, we restrict ourselves to the case $(c_R - c)b < 0$, in which the denominator in equation (6.12) has no poles. Then, equation (6.12) becomes

$$\sigma = -\frac{1}{2h_l}\sum_{n=1}^{2} e^{q_n} Ei(1, q_n), \tag{6.13}$$

where

$$q_n = -g h_l^{-1}(1 + (-1)^n i s), \quad n = 1, 2,$$

and Ei is the integral exponent given by

$$Ei(1, x) = \int_1^\infty \frac{e^{-tx}}{t} dt.$$

Now, we investigate two limiting cases. In the limit $|g| \gg |h_l|$ associated with an uncoated half-space, we have the estimate (for details, see Abramowitz & Stegun 1982)

$$\sigma \sim \frac{1}{g(1+s^2)} = \frac{c_R^2}{(c_R^2 - c^2)(1+s^2)}, \tag{6.14}$$

coinciding with the well-known near-resonant regime of the moving load at $c \to c_R$ (e.g. Cole & Huth 1958).

In the other limit $|g| \ll |h_l|$, we arrive at

$$\sigma \sim -\frac{1}{h_l}\left[\gamma + \ln\left(-\frac{g}{h_l}\sqrt{1+s^2}\right)\right], \tag{6.15}$$

where $\gamma \approx 0.577$ is the Euler's constant, (figure 5*b*). The above formula demonstrates that the presence of a coating does not remove the resonance at $c = c_R$. The reason is that, despite the dispersion of the surface wave due to the influence of the coating, the extremal value of the phase speed is still given by the Rayleigh wave speed. This resonant mechanism is also characteristic of the famous example of a moving load on a beam supported by a Winkler foundation (Timoshenko 1927; also see Timoshenko 1953). For the latter, the critical speed of the load coincides with the minimal phase speed of flexural waves.

The graphs computed by equation (6.13) at $s = 0$ are presented in figure 5. The limiting curves (6.14) and (6.15) in figure 5*a*,*b* are depicted by the solid lines.

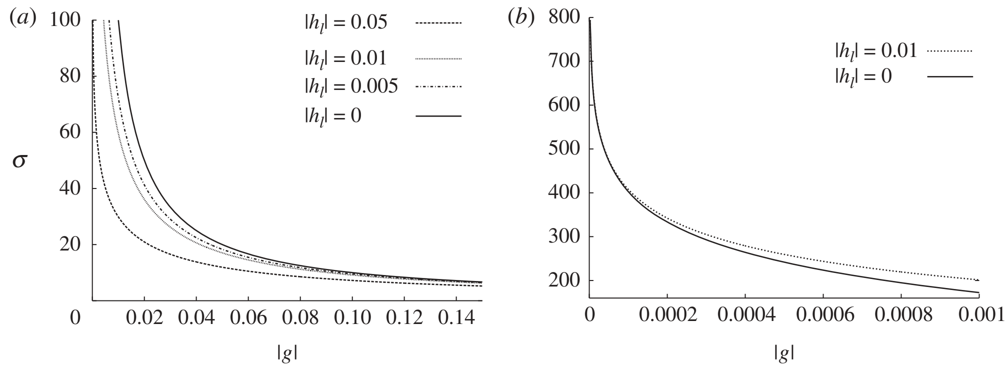

Figure 5. Normal stress σ versus parameter g.

7. Concluding remarks

In this paper, we have not achieved a complete separation of the wave problem on the surface in a differential form as it has been done in Kaplunov *et al.* (2006) for an uncoated elastic half-space. At the same time, the scalar boundary value problem interpreted in §4, as a sort of analogue of a vibrating membrane supported by an incompressible fluid, is still a major simplification in comparison with the original three-dimensional vector hyperbolic system for a coated elastic half-space. In addition, the pseudo-differential equation (5.4) treated as a singular perturbed hyperbolic equation may be analysed using the method of matched asymptotic expansions (see the first example in §6). On the application side, the explicit analytical results obtained for the illustrative examples may be used for non-destructive testing of the coating thickness and the elastic moduli of the substrate.

The derived long-wave model may easily be extended to linear surface waves in case of anisotropic and pre-stressed coatings. The generalizations to nonlinear coatings are less obvious. Nevertheless, there is a possibility of hybrid numeric–analytic approaches combining numerical treatment of nonlinear surface behaviour with analytic formulae for the interior. Moreover, in case of a weak nonlinear coupling, there is a room for asymptotic considerations.

J.K. and D.A.P. thank the City University of Hong Kong for its support and hospitality. The authors also gratefully acknowledge E. Nolde and A. Pichugin for fruitful discussions.

References

Abramowitz, M. & Stegun, I. A. 1982 *Handbook of mathematical functions*. New York, NY: Dover.

Achenbach, J. D. 1998 Explicit solutions for carrier waves supporting surface waves and plate waves. *Wave Motion* **28**, 89–97. (doi:10.1016/S0165-2125(97)00056-5)

Achenbach, J. D. & Keshava, S. P. 1967 Free waves in a plate supported by a semi-infinite continuum. *ASME J. Appl. Mech.* **34**, 397–404.

Auld, B. A. 1990 *Acoustic fields and waves in solids*, 2nd edn. Malabar, FL: Krieger.

Bövik, P. 1996 A comparison between the Tiersten model and $O(h)$ boundary conditions for elastic surface waves guided by thin layers. *ASME J. Appl. Mech.* **63**, 162–167.

Chadwick, P. 1976 Surface and interfacial waves of arbitrary form in isotropic elastic media. *J. Elast.* **6**, 73–80. (doi:10.1007/BF00135177)

Cole, J. D. 1968 *Perturbation methods in applied mathematics.* Waltham, MA: Blaisdell.

Cole, J. & Huth, J. 1958 Stresses produced in a half plane by moving loads. *ASME J. Appl. Mech.* **25**, 433–436.

Dai, H.-H. & Cai, Z. 1999 Uniform asymptotic analysis for waves in an incompressible elastic rod. II. Disturbances superimposed on a pre-stressed state. *IMA J. Appl. Math.* **62**, 1–29. (doi:10.1093/imamat/62.1.1)

Dai, H.-H. & Cai, Z. 2000 Uniform asymptotic analysis for transient waves in a pre-stressed compressible hyperelastic rod. *Acta Mech.* **139**, 201–230. (doi:10.1007/BF01170190)

Emri, I., Kaplunov, J. D. & Nolde E. V. 2001 Analysis of transient waves in thin structures utilizing matched asymptotic expansions. *Acta Mech.* **149**, 55–68. (doi:10.1007/BF01261663)

Friedlander, F. G. 1948 On the total reflection of plane waves. *Q. J. Mech. Appl. Math.* **1**, 376–384. (doi:10.1093/qjmam/1.1.376)

Fu, Y. B. 2007 Linear and nonlinear wave propagation in coated or uncoated elastic half-spaces. In *Waves in nonlinear pre-stressed materials. CISM Courses and Lectures*, vol. 495 (eds M. Destrade & G. Saccomandi), pp. 103–127.

Georgiadis, H. G. & Lykotrafitis, G. 2001 A method based on the Radon transform for three-dimensional elastodynamic problems of moving loads. *J. Elast.* **65**, 87–129. (doi:10.1023/A:1016135605598)

Goldenveizer, A. L., Kaplunov, J. D. & Nolde, E. V. 1993 On Timoshenko–Reissner type theories of plates and shells. *Int. J. Solids Struct.* **30**, 675–694. (doi:10.1016/0020-7683(93)90029-7)

Kaplunov, J. D., Nolde, E. V. & Rogerson, G. A. 2000 A low-frequency model for dynamic motion in pre-stressed incompressible elastic structures. *Proc. R. Soc. Lond. A* **456**, 2589–2610. (doi:10.1098/rspa.2000.0627)

Kaplunov, J., Zakharov A. & Prikazchikov D. A. 2006 Explicit models for elastic and piezoelastic surface waves. *IMA J. Appl. Math.* **71**, 768–782. (doi:10.1093/imamat/hxl012)

Kaplunov, J., Nolde E. & Prikazchikov D. A. In press. A revisit to the moving load problem using an asymptotic model for the Rayleigh wave. *Wave Motion.* (doi:10.1016/j.wavemoti.2010.01.005)

Kovalev, A. S., Mayer, A. P., Eckl, C. & Maugin, G. A. 2002 Solitary Rayleigh waves in the presence of surface nonlinearities. *Phys. Rev. E* **66**, 036615. (doi: 10.1103/PhysRevE.66.036615)

Niklasson, J., Datta, S. K. & Dunn, M. L. 2000 On approximating guided waves in plates with thin anisotropic coating by means of effective boundary conditions. *J. Acoust. Soc. Am.* **108**, 924–933. (doi:10.1121/1.1286882)

Parker, D. F. & Kiselev, A. P. 2009 Rayleigh waves having generalised lateral dependence. *Quart. J. Mech. Appl. Math.* **62**, 19–30. (doi:10.1093/qjmam/hbn022)

Prudnikov, A. P., Brichkov, Y. A. & Marichev, O. I. 1986 *Integrals and series.* New York, NY: Gordon & Breach.

Qian, Z.-H., Jin, F., Lu, RT.-J. & Kishimoto K. 2009 Transverse surface waves in a functionally graded piezoelectric substrate coated with a finite-thickness metal waveguide layer. *Appl. Phys. Lett.* **94**, 023501. (doi:10.1063/1.3070540)

Shuvalov, A. L., & Every, A. G. 2008 On the long-wave onset of dispersion of the surface-wave velocity in coated solids. *Wave Motion* **45**, 857–863. (doi:10.1016/j.wavemoti.2007.12.002)

Steigmann, D. J. & Ogden, R. W. 2007 Surface waves supported by thin-film/substrate interactions. *IMA J. Appl. Math.* **72**, 730–747. (doi:10.1093/imamat/hxm018)

Tiersten, H. F. 1969 Elastic surface waves guided by thin films. *J. Appl. Phys.* **40**, 770–789.

Timoshenko, S. 1927 Method of analysis of statical and dynamical stresses in rail. In *Proc. Second Int. Congress of Appl. Mech., Zurich*, 12–17 September 1926, pp. 1–12. Zurich, Switzerland: Orell Füssli Verlag.

Timoshenko, S. P. 1953 *The collected papers of S. P. Timoshenko*, pp. 422–435. London, UK: McGraw-Hill.

Wang, C.-Y. & Achenbach, J. D. 1996 Lamb's problem for solids of general anisotropy. *Wave Motion* **24**, 227–242. (doi:10.1016/S0165-2125(96)00016-9)

Willis, J. R. 1971 Interfacial stresses induced by arbitrary loading of dissimilar elastic half-spaces joined over a circular region. *J. Inst. Math. Appl.* **7**, 179–197. (doi:10.1093/imamat/7.2.179)